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Convergence Theorems for Modified Mann Reich-Sabach Iteration Scheme for Approximating the Common Solution of Equilibrium Problems and Fixed Point Problems in Hilbert Spaces with Numerical Examples

Felicia. O. Isiogugu*, P. Pillay, C. C. Okeke, F. U. Ogbuisi and P. U.Nwokoro
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Abstract. *It is proved that the modified Reich-Sabach iteration scheme introduced recently by Isiogugu et al. in a real Hilbert space H, converges strongly to a common element of the fixed point sets of a finite family of multi-valued strictly pseudocontractive-type mappings and the set of solutions of a finite family of equilibrium problems. This work is a continuation of the study on the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \| z - u_n \|^2 \leq \| z - x_n \|^2\}$, $x_{n+1} = P_{K_{n+1}} x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunction. The obtained results improve, complement and extend many results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.*

Key Words. *Hilbert spaces, $k-$strictly pseudocontractive-type mapping, strong convergence, strict fixed point sets, equilibrium problem, finite families.*

Felicia. O. Isiogugu*
Department of Mathematics, University of Nigeria, Nsukka, Nigeria
School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Westville Campus, Durban 4000, South Africa
Email: felicia.isiogugu@unn.edu.ng
Orcid id:0000-0002-3959-95 62

P. Pillay
School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Westville Campus, Durban 4000, South Africa
Email: pillaypi@ukzn.ac.za

C. C. Okeke
School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Westville Campus, Durban 4000, South Africa
Email: bueze4christ@yahoo.co\m

F. U. Ogbuisi
Department of Mathematics, University of Nigeria, Nsukka, Nigeria
Email: ferdinard.ogbuisi@unn.edu.ng
Orcid id: 0000-0002-8255-5522

P. U.Nwokoro
Department of Mathematics, University of Nigeria, Nsukka, Nigeria
Email: peter.nwokoro@unn.edu.ng

1.0 Introduction

Let X be a normed space. A subset K of X is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$||x - k|| = \inf\{||x - y|| : y \in K\} = d(x, K) \quad (1)$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of X by $CB(X)$, the family of all nonempty subsets of X by 2^X and the family of all proximinal subsets of X by $P(X)$, for a nonempty set X while H denotes the Hausdorff metric induced by the metric d on a normed space X, that is, for every $A, B \in CB(X)$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Let $T: X \to X$ be a map. A point $x \in X$ is called a fixed point of T if $x = Tx$. If $T: X \to 2^X$ is a multi-valued map from X into the family of nonempty subsets of X, then x is a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, x is called a strict fixed point of T. The set $F(T) = \{x \in D(T): x \in Tx\}$(respectively $F(T) = \{x \in D(T): x = Tx\}$) is called the fixed point set of multi-valued(respectively single-valued) map T while the set $F_S(T) = \{x \in D(T): Tx = \{x\}\}$ is called the strict fixed point set of T.

A multi-valued mapping $T: D(T) \subseteq X \to 2^X$ is called $L-Lipschitzian$ if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx,Ty) \leq L||x-y||. \qquad (2)$$

In (1.2), if $L \in [0,1)$ T is said to be a contraction while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{x \in D(T): x \in Tx\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tx,Tp) \leq ||x-p||. \qquad (3)$$

Clearly, every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. T is said to be k-strictly pseudocontractive-type of Isiogugu [1] if there exists $k \in [0,1)$ such that given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u-v|| \leq H(Tx,Ty)$ and

$$H^2(Tx,Ty) \leq ||x-y||^2 + k||x-u-(y-v)||^2. \qquad (4)$$

If $k = 1$ in (1.4), T is said to be pseudocontractive-type while T is nonexpansive-type if $k = 0$. Every multi-valued nonexpansive mapping $T: D(T) \subseteq X \to P(X)$ is nonexpansive-type. In a real Hilbert space H, $T: D(T) \subseteq H \to CB(H)$ is said to be k-strictly pseudocontractive of Chidume *et al*. (2013) if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$

$$H^2(Tx,Ty) \leq ||x-y||^2 + k||x-u-(y-v)||^2 \qquad (5)$$

for all $u \in Tx$, $v \in Ty$. If $k = 1$, T is said to be pseudocontractive. It is easy to see that every k-strictly pseudocontractive mapping $T: D(T) \subseteq H \to P(H)$ is k-strictly pseudocontractive-type.

A multi-valued map $T: D(T) \subseteq X \to 2^X$. is said to be of type-one (see for example [(Isiogugu *et al*., 2016; Isiogugu *et al*., 2016) if given any pair $x, y \in D(T)$, then

$$||u-v|| \leq H(Tx,Ty), for\ all\ u \in P_T x, v \in P_T y,) \qquad (6)$$

where for each $a \in D(T)$, $P_T a = \{u \in Ta: \| u - a \| = d(a, Ta)\}$. Let H be a real Hilbert space with an inner product $\langle .,. \rangle$ and a norm $\|.\|$, respectively and let K be a nonempty closed convex subset of H. Let $A: H \to H$ be an operator on H and $F: K \times K \to \mathbb{R}$ be a bifunction on K, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of A in K denoted by $VIP(A,K)$ is to find an $x^* \in K$ such that

$$\langle x - x^*, A(x^*) \rangle \geq 0, \quad \forall x \in K, \qquad (7)$$

while the equilibrium problem for F is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in K. \qquad (8)$$

The set of solutions of (1.8) is denoted by $EP(F)$. Suppose $F(x,y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if w is a solution of (7). Many problems in optimization, economics and physics reduce to finding a solution of (1.7), (see for examples, (Isiogugu *et al*., 2011; Blum and Oettli, 1994 Combettes and Hirstoaga, 2005; Takahashi and Zembayash, 2008;Moudafi, 2003) and the references therein. The following conditions are assumed for solving the equilibrium problems for a bifunction $F: K \times K \to \mathbb{R}$,

(A1) $F(x,x) = 0$ for all $x \in K$.

(A2) F is monotone, that is , $F(x,y) + F(y,x) \leq 0$, for all $x, y \in K$.

(A3) For each $x, y, z \in K$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$.

(A4) For each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Several algorithms have been introduced by authors for approximating solutions of equilibrium problems for a bifunction (or finite family of bifunctions) as well as a common element of the fixed point sets of finite family of multi-valued (or single-valued) mappings and the set of solutions of finite family of equilibrium problems (see for examples (Isiogugu, 2016; Reich and Sabach, 2012; Tada and Takahashi, 2008; Qin and Wang, 2011; Jarboon and Kuman, 2010) and references therein).

Reich and Sabach (2012) proposed three algorithms for solving (common) equilibrium problems of bifunction(s) g in a general reflexive Banach spaces using the well chosen convex function f, the Bregman distance and the projection associated with it. They proposed one of the algorithms as follows.

Let X be a reflexive Banach space, $\{K_i\}_{i=1}^{N}$ a finite family of nonempty, closed and convex subsets of X. Let $\{\lambda^i\}_{i=1}^{N}$ be a finite family of positive real numbers and $\{g_i\}_{i=1}^{N}$ a finite family of bifunctions, with $g_i: K_i \times K_i \to \mathbb{R}$ for each $i = 1,2,\ldots,N$. Suppose $f: X \to R$ is a coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of X, $Res^{f}_{\lambda^i g_i}$ is the resolvent of g_i with respect to λ^i and f for each $i = 1,2,\ldots,N$ and D_f is the Bregman distance on X. If $E = \bigcap_{i=1}^{N} EP(g_i) \neq \emptyset$, then the sequences $\{x_n\}_{n=1}^{\infty}$ were generated from an arbitrary $x_0 \in X$ as follows.

Algorithm 1 (Algorithm II [8]).

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \qquad i = 1,2,\ldots,N \\ y_n^i = Res^{f}_{\lambda_n^i g_i}(x_n + e_n^i), \\ K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ K_{n+1} = \bigcap_{i=1}^{N} K_{n+1}^i, \\ x_{n+1} = proj^{f}_{K_{n+1}}(x_0), \qquad n = 0,1,2,\ldots. \end{cases} \tag{9}$$

(9)

Furthermore, they proved that if $\liminf_{n\to\infty} \lambda_n^i > 0$ and $\lim_{n\to\infty} e_n = 0$, the sequences converge strongly to $proj^{f}_{E}(x_0)$.

Recently, Isiogugu (2016) obtained a strong convergence of a hybrid algorithm to a common element of the fixed point sets of two multi-valued strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces using a strict fixed point set condition. She proved the following theorem:

Theorem 1 ([9]). Let K be a nonempty closed convex subset of a real Hilbert space H, let $f: K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $S, T: K \to P(K)$ be two strictly pseudocontractive-type mappings with contractive coefficients λ_1 and λ_2 respectively such that $\mathbb{F} = F_s(T) \cap \ F_s(S) \cap \ EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in K$ as follows:

$$\begin{cases} x_0 \in H, \\ K_1 = K, \\ x_1 = P_K x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)[\beta_n v_n + (1-\beta_n) z_n], \\ u_n \in K \ such \ that \ f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\ K_{n+1} = \{z \in K_n : \| z - u_n \|^2 \leq \| z - x_n \|^2\}, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases} \tag{10}$$

where $v_n \in Tx_n$, $z_n \in Sx_n$. $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1] satisfying

(i) $\alpha_n \geq \max\{\lambda_1, \lambda_2\}$.

(ii) $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)(\alpha_n - \lambda_1) > 0$, $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - \lambda_2)\beta_n > 0$

(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$.

Isiogugu *et al.* (2016) observed that in Algorithms 1, if $T^i: K^i \to K^i$ is the identity map on K^i, $i = 1,2,\ldots,N$, respectively and $v_n^i = \alpha_n x_n + (1-\alpha_n) T^i x_n + e_n^i = x_n + e_n^i$, then we can rewrite Algorithm 1 in the following form:

Algorithm 2.

$$
\begin{cases}
x_0 \in X, \\
K_0^i = X, \qquad i = 1,2,\ldots,N \\
v_n^i = \alpha_n x_n + (1-\alpha_n)T^i x_n + e_n^i, \\
y_n^i = Res^f_{\lambda_n^i g_i}(v_n^i), \\
K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \le D_f(z, v_n^i)\}, \\
K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\
x_{n+1} = proj^f_{K_{n+1}}(x_0), \qquad n = 0,1,2,\ldots.
\end{cases} \tag{11}
$$

Motivated by the above observations in Algorithms 2 and the iteration scheme in Theorem 1 considered by Isiogugu in [9], Isiogugu *et al*. (2016) constructed a hybrid algorithm for approximating a common element of the fixed point sets of a finite family of multi-valued nonexpansive mappings and the set of solutions of a finite family of equilibrium problems in Hilbert spaces without error terms. They studied the following iteration scheme: Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of nonexpansive mappings such that $f^i: K^i \times K^i \to \mathbb{R}$ and $T^i: K^i \to P(K^i)$ for all $i = 1,2,\ldots,N$ respectively. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in $[0,1]$ and $\{r_n^i\}_{n=1}^\infty \subset [a,\infty)$ for some $a > 0$ for all i=1, 2, ..., N, then from an arbitrary $x_0 \in H$ the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows.

Algorithm 3(Algorithm 7 (*Isiogugu et al.,* 2016**).**

$$
\begin{cases}
x_0 \in H, \\
K_1^i = K^i, \quad \forall\ i = 1,2,\ldots,N, \\
K_1 = \bigcap_{i=1}^N K_1^i, \\
x_1 = P_{K_1} x_0, \\
y_n^i = \alpha_n^i x_n + (1-\alpha_n^i) v_n^i, \\
u_n^i \in K^i \ \ such\ that\ \ f^i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \ge 0, \quad \forall y \in K^i, \\
K_{n+1}^i = \{z \in K_n^i : \| z - u_n^i \| \le \| z - x_n \|\}, \\
K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\
x_{n+1} = P_{K_{n+1}} x_0,
\end{cases} \tag{12}
$$

where $v_n^i \in T^i x_n$.

Using the above algorithm, they proved the following theorem:

Theorem 2(Theorem 2 [Isiogugu *et al.,* 2016**).** Let H, $\{K^i\}_{i=1}^N$, $\{T^i\}_{i=1}^N$, $\{f^i\}_{i=1}^N$, $\{\alpha_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ be as in algorithm 3. Suppose f^i satisfying (A1)-(A4) for all $i = 1,2,\ldots,N$, $\mathbb{F} = \bigcap_{i=1}^N F_s(T^i) \cap \left(\bigcap_{i=1}^N EP(f^i)\right) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1,2,\ldots,N$ and for all $n \ge 1$, $\liminf_{n\to\infty} \alpha_n^i(1-\alpha_n^i) > 0$.

The aim of this research is to establish first, that if $\{T^i\}_{i=1}^N$ is a finite family of multi-valued k^i-strictly pseudocontractive-type mappings, the results of Theorem 2 is true under some mild conditions on the sequences $\{\alpha_n^i\}_{n=1}^\infty$ and contraction coefficients k^i 's, $I = 1,2,\ldots,N$. Second, the numerical computability of the algorithm under the conditions on the sequences $\{\alpha_n^i\}_{n=1}^\infty$ and contraction coefficients k^i 's, $I = 1,2,\ldots,N$ for the case namely: (i) $F_s(T^i) \neq \emptyset$; (ii) $F_s(T^i) = \emptyset$ but T^i is of type-one. The results of this research are great contributions towards the resolution of the controversy over the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \| z - u_n \|^2 \le \| z - x_n \|^2\}$, $x_{n+1} = P_{K_{n+1}} x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^\infty$ is the sequence of the resolvent of the bifunctions. They also generalize, extend, complement and improve the result considered in (Isiogugu, 2016; Isiogugu *et al.*, 2016) as well as some recent results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.

2.0 Preliminaries

Lemma 1 ([1]). Let H be a real Hilbert space, and let $T: D(T) \subseteq H \to 2^H$ be a k-strictly pseudocontractive-type mapping, then T is an L-Lipschitzian.

Lemma 2 ([9]). Let K be a nonempty subset of a real Hilbert space H and let $T: K \to P(K)$ be a k-strictly pseudocontractive-type mapping such that $F_s(T)$ is nonempty. Then $F_s(T)$ is closed and convex.

Lemma 3: Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let P_K be the convex projection onto K. Then, convex projection is characterized by the following relations;

(i) $x^* = P_K(x) \Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.

(ii) $||x - P_K x||^2 \leq ||x - y||^2 - ||y - P_K x||^2$.

(iii)$||x - P_K y||^2 \leq ||x - y||^2 - ||P_K y - y||^2$.

Lemma 4 ([4]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F: K \times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$

Lemma 5 ([5]). Let K be a nonempty closed convex subset of a real Hilbert space H. Assume that $F: K \times K \to \mathbb{R}$ that satisfies (A1)-(A4). Let $r > 0$ and $x \in H$, define $T_r: H \to 2^K$ by

$$T_r(x) = \{z \in K: F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0\}, \quad \forall y \in K.$$

Then the following conditions hold:

(1) T_r is single valued.

(2) T_r is firmly nonexpansive, that is for any $x, y \in H$, $\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

Lemma 6 ([6]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F: K \times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then for all $x \in H$ and $p \in F(T_r)$

$$\| p - T_r x \|^2 + \| T_r x - x \|^2 \leq \| p - x \|^2.$$

3.0 Main Results

We now consider the following algorithm.

Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of k^i- strictly pseudocontractive-type mappings such that $f^i: K^i \times K^i \to \mathbb{R}$ and $T^i: K^i \to P(K^i)$ for all $i = 1,2, \ldots, N$ respectively. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$ for all i=1, 2, ..., N, then from an arbitrary $x_0 \in H$ we generate the sequence $\{x_n\}_{n=1}^\infty$ as follows.

Algorithm 4.

$$\begin{cases} K_1^i = K^i, \quad \forall\ i = 1,2, \ldots, N, \\ y_n^i = \alpha_n^i x_n + (1 - \alpha_n^i) v_n^i, \\ u_n^i \in K^i \ such\ that\ f^i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i: \| z - u_n^i \|^2 \leq \| z - x_n \|^2\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases} \tag{13}$$

where $v_n^i \in T^i x_n$.

Theorem 3. Let H, $\{K^i\}_{i=1}^N$, $\{T^i\}_{i=1}^N$, $\{f^i\}_{i=1}^N$, $\{\alpha_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ be as in algorithm 4. Suppose f^i satisfying (A1)-(A4) for all $i = 1,2, \ldots, N$, $\mathbb{F} = \bigcap_{i=1}^N F_s(T^i) \cap \ (\bigcap_{i=1}^N EP(f^i)) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1,2, \ldots, N$ and for all $n \geq 1$,

(i) $\alpha_n^i > k_i$, (ii) $\liminf_{n\to\infty}(1 - \alpha_n^i)(\alpha_n^i - k^i) > 0$.

Proof. Since K_n^i is closed and convex for all $n \geq 1$ and for all $i = 1,2, \ldots, N$, $K_n = \bigcap_{i=1}^N K_n^i$ is closed and convex and hence $P_{K_{n+1}} x_0$ is well defined, also, $u_n^i = T_{r_n^i} y_n^i$. Next we show that $\mathbb{F} \subset K_n$, for all $n \geq 1$. $\mathbb{F} \subset K_1^i = K^i$ for all $i = 1,2, \ldots, N$, therefore, $\mathbb{F} \subset \bigcap_{i=1}^N K_1^i = K_1$. Assume $\mathbb{F} \subset K_k = \bigcap_{i=1}^N K_k^i$. Using Lemma 5, for all $q \in \mathbb{F}$ we have $\| q - u_k^i \|^2 = \| q - T_{r_k^i} y_k^i \|^2 \leq \| q - y_k^i \|^2$

$= \alpha_k^i \| x_k - q \|^2 + (1 - \alpha_k^i) \| v_k^i - q \|^2 - \alpha_k^i(1 - \alpha_k^i) \| x_k - v_k^i \|^2 \leq \alpha_k^i \| x_k - q \|^2 + (1 - \alpha_k^i) H^2(T^i x_k, T^i q) - \alpha_k^i(1 - \alpha_k^i) \| x_k - v_k^i \|^2$

$$\leq \alpha_k^i \parallel x_k - q \parallel^2 + (1-\alpha_k^i)[\parallel x_k - q \parallel^2 + k^i \parallel x_k - v_k^i \parallel^2] - \alpha_k^i(1-\alpha_k^i) \parallel x_k - v_k^i \parallel^2 = \parallel x_k - q \parallel^2 - (1-\alpha_k^i)(\alpha_k^i - k^i) \parallel x_k - v_k^i \parallel^2 \quad (14)$$

$$\leq \parallel x_k - q \parallel^2.$$

This shows that $q \in K_{k+1}^i$ for all $i = 1,2,\ldots,N$, therefore, $q \in \bigcap_{i=1}^N K_{k+1}^i = K_{k+1}$ and hence $\mathbb{F} \subseteq K_n$ for all $n \geq 1$. From $x_n = P_{K_n}x_0$ and Lemma 3(i), we obtain

$$\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall\ y \in K_n. \quad (15)$$

and

$$\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall\ q \in F. \quad (16)$$

Using Lemma 3(ii), we have that

$$\parallel x_n - x_0 \parallel^2 = \parallel P_{K_n}x_0 - x_0 \parallel^2 \leq \parallel x_0 - q \parallel^2 - \parallel q - x_n \parallel^2 \leq \parallel x_0 - q \parallel^2,$$

for each $q \in \mathbb{F} \subset K_n$ and for all $n \geq 1$. Consequently the sequences $\{x_n\}$, $\{v_n^i\}$, $i = 1,2,\ldots,N$ are bounded. Furthermore, since $x_n = P_{K_n}x_0$, $x_{n+1} = P_{K_{n+1}}x_0 \in K_{n+1} \subset K_n$ then from definition of P_K we have $\parallel x_n - x_0 \parallel \leq \parallel x_{n+1} - x_0 \parallel$ for all $n \geq 1$. Therefore the sequence $\{\parallel x_n - x_0 \parallel\}_{n=1}^\infty$ is nondecreasing. Thus, $\lim_{n\to\infty} \parallel x_n - x_0 \parallel$ exists. From the construction of K_n we have that $K_m \subset K_n$ and $x_m = P_{K_m}x_0 \in K_n$ for any integer $m \geq n$. Thus, from Lemma 3(iii)

$$\parallel x_m - x_n \parallel^2 = \parallel x_m - P_{K_n}x_0 \parallel^2 \leq \parallel x_m - x_0 \parallel^2 - \parallel x_n - x_0 \parallel^2. \quad (17)$$

Letting $m, n \to \infty$ in (2.5), we have $\parallel x_m - x_n \parallel \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since H is Hilbert and K^i is closed and convex for all $i = 1,2,\ldots,N$, we can assume that $x_n \to p \in K^i$, for all $i = 1,2,\ldots,N$ as $n \to \infty$. We now show that $p \in F(T^i)$, for all $i = 1,2,\ldots,N$. In particular when $m = n+1$ in (2.5) we obtain

$$\lim_{n\to\infty} \parallel x_{n+1} - x_n \parallel = 0. \quad (18)$$

Also, since $x_{n+1} \in K_{n+1}$, $\parallel x_{n+1} - u_n^i \parallel \leq \parallel x_{n+1} - x_n \parallel$, using (2.6),

$$\lim_{n\to\infty} \parallel x_{n+1} - u_n^i \parallel = 0. \quad (19)$$

Combining (2.6) and (2.7) we have

$$\lim_{n\to\infty} \parallel x_n - u_n^i \parallel = 0. \quad (20)$$

It follows from $\lim_{n\to\infty} \parallel x_n - p \parallel = 0$ and (2.8) that

$$\operatorname{Lim}_{n\to\infty} \parallel u_n^i - p \parallel = 0. \quad (21)$$

Setting $n = k$ in (2.2), we obtain

$$\parallel u_n^i - q \parallel^2 \leq \parallel x_n - q \parallel^2 - (1-\alpha_n^i)(\alpha_n^i - k^i) \parallel x_n - v_n^i \parallel^2. \quad (22)$$

Observe that

$$\parallel q - x_n \parallel^2 - \parallel q - u_n^i \parallel^2 = \parallel x_n \parallel^2 - \parallel u_n^i \parallel^2 - 2\langle q, x_n - u_n^i \rangle$$

$$\leq \parallel x_n - u_n^i \parallel (\parallel x_n \parallel + \parallel u_n^i \parallel) + 2 \parallel q \parallel \parallel x_n - u_n^i \parallel.$$

It follows from (20) that

$$\lim_{n\to\infty} \parallel q - x_n \parallel - \parallel q - u_n^i \parallel = 0. \quad (23)$$

Using $\liminf_{n\to\infty}(1-\alpha_n^i)(\alpha_n^i - k^i) > 0$ we deduce from (2.10) that $\lim_{n\to\infty} \parallel x_n - v_n^i \parallel = 0$. Hence $p \in F(T^i)$ for each $i = 1,2,\ldots,N$. It then follows that $p \in \bigcap_{i=1}^N F(T^i)$. It remains to show that p is in $EP(f^i)$ for all $i = 1,2,\ldots,N$. Now from (22)

$$\parallel q - y_n^i \parallel \leq \parallel q - x_n \parallel. \quad (24)$$

Also, using $u_n^i = T_{r_n^i}y_n^i$, Lemma 6 and (2.12) we have

$$\parallel u_n^i - y_n^i \parallel^2 = \parallel T_{r_n^i}y_n^i - y_n^i \parallel^2$$

$$\leq \parallel q - y_n^i \parallel^2 - \parallel q - T_{r_n^i}y_n^i \parallel^2$$

$$\leq \parallel q - x_n \parallel^2 - \parallel q - T_{r_n^i}y_n^i \parallel^2$$

$$= \parallel q - x_n^i \parallel^2 - \parallel q - u_n^i \parallel^2. \quad (25)$$

Therefore, from (23) and (25)

$$\lim_{n\to\infty} \parallel u_n^i - y_n^i \parallel = 0. \quad (26)$$

Consequently, from (21) and (26)

$$\lim_{n\to\infty} \parallel y_n^i - p \parallel = 0. \quad (27)$$

From the assumption that $r_n^i \geq a > 0$,

$$\lim_{n\to\infty} \frac{\parallel u_n^i - y_n^i \parallel}{r_n^i} = 0. \quad (28)$$

Since $u_n^i = T_{r_n^i}y_n^i$ implies

$$f(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq 0,$$

we deduce from (A2) that

$$\frac{\parallel u_n^i - y_n^i \parallel^2}{r_n^i} \geq \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq -f(u_n^i, y) \geq f(y, u_n^i).\ \forall y \in K^i$$

By taking limit as $n \to \infty$ of the above inequality and from (A4), (21) and (27) $f(y,p) \leq 0$, for all $y \in K^i$. Let $t \in (0,1)$ and for all $y \in K^i$, since $p \in K^i$, $y_t = ty + (1-t)p \in K^i$. Hence $f(y_t, p) \leq 0$. Therefore, from (A1),

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y),$$

that is, $f(y_t, y) \geq 0$. Letting $t \downarrow 0$, from (A3) we obtain $f(p,y) \geq 0$ for all $y \in K^i$ so that $p \in EP(f^i)$ for all $i = 1,2,\ldots,N$. Hence $p \in \mathbb{F}$.

Finally, we show that $p = P_{\mathbb{F}}x_0$. By taking the limits as $n \to \infty$ in (2.3) we have

$$\langle p - q, x_0 - p \rangle \geq 0, \quad \forall q \in \mathbb{F}.$$

Thus, from Lemma 3(i) $p = P_{\mathbb{F}}x_0$. This completes the proof.

Corollary 1 (Theorem 2 Isiogugu *et* al., 2016)]). Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of nonexpansive-type mappings such that $f^i: K^i \times K^i \to \mathbb{R}$ and $T^i: K^i \to P(K^i)$ for all $i = 1,2,\ldots,N$ respectively. Suppose f^i satisfying (A1)-(A4) for all $i = 1,2,\ldots,N$, and $\mathbb{F} = \bigcap_{i=1}^N F(T^i) \cap (\bigcap_{i=1}^N EP(f^i)) \neq \emptyset$. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a,\infty)$ for some $a > 0$ for all i=1, 2, ..., N, then the sequence $\{x_n\}_{n=1}^\infty$ generated from an arbitrary $x_0 \in H$ as follows,

$$\begin{cases} K_1^i = K_i, \quad for\ all\ i = 1,2,\ldots,N, \\ y_n^i = \alpha_n x_n + (1-\alpha_n)v_n^i, \\ u_n^i \in K^i\ such\ that\ f^i(u_n^i, y) + \dfrac{1}{r_n}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \| z - u_n^i \|^2 \leq \| z - x_n \|^2\}, \\ K_{n+1} = \displaystyle\bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}}x_0, \end{cases}$$

where $v_n^i \in P_{T^i}x_n$, converges strongly to $p \in P_{\mathbb{F}}x_0$ if for each $i = 1,2,\ldots,N$ and for all $n \geq 1$, $\liminf_{n\to\infty} \alpha_n^i(1-\alpha_n^i) > 0$.

Proof. The proof follows easily from Theorem 3.

Theorem 4. Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of type-one k^i −strictly pseudocontractive mappings such that $f^i: K^i \times K^i \to \mathbb{R}$ and $T^i: K^i \to P(K^i)$ for all $i = 1,2,\ldots,N$ respectively. Suppose f^i satisfying (A1)-(A4) for all $i = 1,2,\ldots,N$, and $\mathbb{F} = \bigcap_{i=1}^N F(T^i) \cap (\bigcap_{i=1}^N EP(f^i)) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in [0,1] and $\{r_n\}_{n=1}^\infty \subset [a,\infty)$ for some $a > 0$, then the sequence $\{x_n\}_{n=1}^\infty$ generated from an arbitrary $x_0 \in H$ as follows,

$$\begin{cases} K_1^i = K_i, \quad for\ all\ i = 1,2,\ldots,N, \\ y_n^i = \alpha_n x_n + (1-\alpha_n)v_n^i, \\ u_n^i \in K^i\ such\ that\ f^i(u_n^i, y) + \dfrac{1}{r_n}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \| z - u_n^i \|^2 \leq \| z - x_n \|^2\}, \\ K_{n+1} = \displaystyle\bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}}x_0, \end{cases}$$

where $v_n^i \in P_{T^i}x_n$, converges strongly to $p \in P_{\mathbb{F}}x_0$ if for each $i = 1,2,\ldots,N$ and for all $n \geq 1$,
(i) $\alpha_n > k^i$ (ii) $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - k^i) > 0$.

Proof. The proof is a similar procedure to the proof of Theorem 3, therefore, it is omitted.

Corollary 2. Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of type-one nonexpansive-type mappings such that $f^i: K^i \times K^i \to \mathbb{R}$ and $T^i: K^i \to P(K^i)$ for all $i = 1,2,\ldots,N$ respectively. Suppose f^i satisfying (A1)-(A4) for all $i = 1,2,\ldots,N$, and $\mathbb{F} = \bigcap_{i=1}^N F(T^i) \cap (\bigcap_{i=1}^N EP(f^i)) \neq \emptyset$. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a,\infty)$ for some $a > 0$ for all i=1, 2, ..., N, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_0 \in H$ as follows,

$$\begin{cases} K_1^i = K_i, \quad for\ all\ i = 1,2,\ldots,N, \\ y_n^i = \alpha_n x_n + (1-\alpha_n)v_n^i, \\ u_n^i \in K^i\ such\ that\ f^i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \| z - u_n^i \|^2 \leq \| z - x_n \|^2\}, \\ K_{n+1} = \bigcap_{i=1}^{N} K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n^i \in P_{T^i} x_n$, converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1,2,\ldots,N$ and for all $n \geq 1$, $\liminf_{n\to\infty} \alpha_n^i (1-\alpha_n^i) > 0$.

Proof. The proof follows easily from Theorem 4.

4.0 Numerical Computations

In our numerical computation, we shall consider the following cases :

(i) the common element is a strict fixed point of each T^i (that is Theorem 3).

(ii) the common element is not a strict fixed point for any T^i but each T^i is of type-one (that is Theorem 4).

Case i.

Example 1. Let $H = \mathbb{R}$ (the reals with the usual metric and inner product), $i = 1,2,\ldots,4$ and $K^i = [-4,10i]$, for all i. Then for each i, we define:

(i) $T^i : [-4,10i] \to P([-4,10i])$ by

$$T^i x = \begin{cases} [-2ix, -\frac{9ix}{4}], & x \in [-4,0] \\ \{-\frac{3x}{9i}\}, & x \in (0,10i]. \end{cases}$$

we have that for any $x, y \in [-4,0]$,

$$H^2(T^i x, T^i y) = \| \tfrac{9i}{4}(x-y) \|^2 = (\tfrac{9i}{4})^2 \| x - y \|^2 = \| x-y \|^2 + ((\tfrac{9i}{4})^2 - 1) \| x-y \|^2.$$

Also, given any $u \in T^i x$, $u = -\alpha x$, $2i \leq \alpha \leq \frac{9i}{4}$ and we can choose $v = -\alpha y \in T^i y$ so that

$$\| x - u - (y - v) \|^2 = (1+\alpha)^2 \| x - y \|^2.$$

It then follows that

$$H^2(T^i x, T^i y) = \| x - y \|^2 + \frac{(9i)^2 - 16}{16(1+\alpha)^2} \| x - u - (y-v) \|^2$$
$$\leq \| x - y \|^2 + \frac{(9i)^2-16}{16(1+2i)^2} \| x - u - (y - v) \|^2, \forall i = 1,2,3,4.$$

Similarly, for any $x \in [-4,0]$, $y \in (0,10i]$,

$$(T^i x, T^i y) = \| \frac{9i}{4}x - \frac{3y}{9i} \|^2 \leq \| \frac{9i}{4}x - \frac{9i}{4}y \|^2 \leq \| x - y \|^2 + \frac{(9i)^2-16}{16(1+2i)^2} \| x - u - (y-v) \|^2, \forall i = 1,2,3,4.$$

Furthermore, for any $x, y \in (0,10i]$,

$$H^2(T^i x, T^i y) = \frac{3}{9i} \| x - y \|^2 \leq \| x - y \|^2 + \frac{(9i)^2-16}{16(1+2i)^2} \| x - u - (y - v) \|^2,$$

for all $i = 1,2,3,4$.

(ii)

$$v_n^i = \begin{cases} -\frac{9ix_n}{4}, & x_n \in [-4,0] \\ -\frac{3x_n}{9i}, & x_n \in (0,10i]. \end{cases}$$

(iii) $\{\alpha_n^i\}_{n=1}^{\infty} = \frac{((9i)^2 - 16 + 16(1+2i)^2)(n + 32(1+2i)^2)^2 + 32(1+2i)^2}{32(1+2i)^2(n+32(1+2i)^2)^2}$,

while $f^i : R \times R \to R$, $\{r_n^i\}_{n=1}^{\infty}$ and $\{u_n^i\}_{n=1}^{\infty}$ are defined as in (Isiogugu *et al.*, 2016). That is,

(iv) $f^i(x,y) = -ix^2 + iy^2$,

Observe that

$$f^i(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \Rightarrow iy^2 - iz^2 + \frac{1}{r}(y-z)(z-x) \geq 0,$$
$$\Rightarrow iy^2 - iz^2 + \frac{1}{r}[yz - xy - z^2 + xz] \geq 0,$$
$$\Rightarrow iry^2 - irz^2 + yz - xy - z^2 + xz \geq 0,$$
$$\Rightarrow iry^2 + (z-x)y - irz^2 - z^2 + xz \geq 0.$$

Now $F(y) = iry^2 + (z-x)y - irz^2 - z^2 + xz$ a is a quadratic function of y with coefficients $a =$

ir , $b = z - x$ and $c = -irz^2 - z^2 + xz$. as follows:

Therefore, we can compute the discriminant Δ of F

$$\begin{aligned}\Delta &= (z-x)^2 + 4ir(irz^2 + z^2 - xz) = z^2 + x^2 - 2xz + 4i^2r^2z^2 + 4irz^2 - 4irxz \\ &= (1 + 4i^2r^2 + 4ir)z^2 - 2(2ir+1)xz + x^2 \\ &= (1+2ir)^2z^2 - 2(1+2ir)xz + x^2 \\ &= [(1+2ir)z - x]^2 \qquad (29)\end{aligned}$$

Obviously, $F(y) \geq 0$ for all $y \in \mathbb{R}$ if it has at most one solution in $\mathbb{R}$. Thus $\Delta \leq 0$ and hence $z = T_{r_n^i}(x) = \frac{x}{1+2ir}$. Consequently

(iv) $\{u_n^i\}_{n=1}^{\infty} = T_{r_n^i}(y_n^i)\{\frac{y_n^i}{2ir_n^i+1}\}_{n=1}^{\infty}$.

(v) $\{r_n^i\}_{n=1}^{\infty} = \{\frac{ni+1}{ni}\}_{n=1}^{\infty}$,

It is easy to see that given any pair $x, y \in [-4, 0]$, we have that $H(T^ix, T^iy) = \frac{9i}{4} \parallel x - y \parallel$, for each T^i, therefore T^i is not nonexpansive for all $i = 1,2,3,4$. Furthermore, $F_s(T^i) = \{0\} \neq \emptyset$, $EPf^i = \{0\}$ for each i and

$$\mathbb{F} = \cap_{i=1}^{N} F_s(T^i) \cap \ (\cap_{i=1}^{N} EP(f^i)) = \{0\}.$$

Also, by method of computation similar to that used by Isiogugu (2013) in, we have T^i is $\frac{(9i)^2-16}{16(1+2i)^2}$ −strictly pseudocontractive-type for each i. The algorithms are computed with Microsoft word Excel 97-2003 Workbook. Table 1 below shows the sequences $\{x_n\}$ and $\{K_n\}$ generated from our computation using two different values of $x_0 = 3, -3$ while Table 2 shows different sequences generated for different values of x_0. In particular, we considered without loss of generality $x_0 = 4, -4, 2, -2, 8, -1$.

Table 1. The values of x_n **and** K_n **for** $x_0 = 3, -3$

n	x_n	K_{n+1}	x_n	K_{n+1}
0	3	[-4, 10]	-3	[-4,10]
1	3	[-4, 1.635451761]	-3	[-1.532563756, 10]
2	1.635451761	[-4, 0.898951648]	-1.532563756	[-0.787076007, 10]
3	0.898951648	[-4, 0.49566239]	-0.787076007	[-0.405188015, 10]
4	0.49566239	[-4, 0.273744239]	-0.405188015	[-0.208877029, 10]
5	0.273744239	[-4, 0.151335613]	-0.208877029	[-0.107772707, 10]
6	0.151335613	[-4, 0.083720857]	-0.107772707	[-0.055641074, 10]
7	0.083720857	[-4, 0.046338328]	-0.055641074	[-0.028739594, 10]
8	0.046338328	[-4, 0.025657188]	-0.028739594	[-0.014849721, 10]
9	0.025657188	[-4, 0.014210344]	-0.014849721	[-0.007674974, 10]
10	0.014210344	[-4, 0.007872308]	-0.007674974	[-0.003967653, 10]
11	0.007872308	[-4, 0.004361977]	-0.003967653	[-0.002051501, 10]
12	0.004361977	[-4, 0.002417323]	-0.002051501	[-0.001060909, 10]
13	0.002417323	[-4, 0.001339817]	-0.001060909	[-0.00054871, 10]
14	0.001339817	[-4, 0.000742689]	-0.00054871	[-0.00028383, 10]
15	0.000742689	[-4, 0.00041173]	-0.00028383	[-0.000146831, 10]

16	0.00041173	[-4, 0.000228274]	-0.000146831	[-0.000075965, 10]
17	0.000228274	[-4, 0.000126571]	-0.000075965	[-0.000039304, 10]
18	0.000126571	[-4, 0.000070184]	-0.000039304	[-0.000020337, 10]
19	0.000070184	[-4, 0.00003892]	-0.000020337	[-0.000010523, 10]
20	0.00003892	[-4, 0.000021584]	-0.000010523	[-0.000005445, 10]
21	0.000021584	[-4, 0.00001197]	-0.000005445	[-0.000002817, 10]
22	0.00001197	[-4, 0.000006638]	-0.000002817	[-0.000001457, 10]
23	0.000006638	[-4, 0.000003681]	-0.000001457	[-0.000000754, 10]
24	0.000003681	[-4, 0.000002041]	-0.000000754	[-0.00000039, 10]
25	0.000002041	[-4, 0.000001132]	-0.00000039	[-0.000000201, 10]
26	0.000001132	[-4, 0.000000627]	-0.000000201	[-0.000000103, 10]
27	0.000000627	[-4, 0.000000347]	-0.000000103	[-0.000000053, 10]
28	0.000000347	[-4, 0.000000192]	-0.000000053	[-0.000000027, 10]
29	0.000000192	[-4, 0.000000106]	-0.000000027	[-0.000000013, 10]
30	0.000000106	[-4, 0.000000058]	-0.000000013	[-0.000000006, 10]
31	0.000000058	[-4, 0.000000032]	-0.000000006	[-0.000000003, 10]
32	0.000000032	[-4, 0.000000017]	-0.000000003	[-0.000000001, 10]
33	0.000000017	[-4, 0.000000009]	-0.000000001	[0, 10]
34	0.000000009	[-4, 0.000000004]	0	
35	0.000000004	[-4, 0.000000002]		
36	0.000000002	[-4, 0.000000001]		
37	0.000000001	[-4, 0]		
38	0			

Table2. The values of x_n **for** $x_0 = 4, -4, 2, -2, 8, -1.$

n	x_n	x_n	x_n	x_n	x_n	x_n
0	4	-4	2	-2	8	-1
1	4	-4	2	-2	8	-1
2	2.180602348	-2.043418342	1.090301174	-1.021709171	4.361204697	-0.510854585
3	1.198602198	-1.049434676	0.599301099	-0.524717338	2.397204396	-0.262358668
4	0.660883187	-0.540250686	0.330441593	-0.270125343	1.321766375	-0.135062671
5	0.36499232	-0.278502706	0.182496159	-0.139251353	0.72998464	-0.069625676
6	0.201780818	-0.143696944	0.100890408	-0.071848472	0.403561637	-0.035924235
7	0.11162781	-0.074188099	0.055813904	-0.037094049	0.223255621	-0.018547024
8	0.061784438	-0.038319459	0.030892218	-0.019159729	0.123568878	-0.009579864

9	0.034209585	-0.019799629	0.017104792	-0.009899814	0.068419172	-0.004949906
10	0.018947126	-0.010233299	0.009473562	-0.005116649	0.037894253	-0.002558324
11	0.010496412	-0.005290205	0.005248205	-0.002645102	0.020992824	-0.00132255
12	0.00581597	-0.002735336	0.002907984	-0.001367667	0.011631941	-0.000683833
13	0.003223098	-0.001414547	0.001611548	-0.000707272	0.006446197	-0.000353636
14	0.001786423	-0.000731615	0.000893211	-0.000365806	0.003572847	-0.000182903
15	0.000990253	-0.000378441	0.000495126	-0.000189219	0.001980506	-0.000094609
16	0.000548974	-0.000195775	0.000274487	-0.000097886	0.001097949	-0.000048943
17	0.000304366	-0.000101287	0.000152183	-0.000050642	0.000608733	-0.000025321
18	0.000168762	-0.000052406	0.000084381	-0.000026202	0.000337525	-0.000013101
19	0.00009358	-0.000027117	0.00004679	-0.000013558	0.000187161	-0.000006779
20	0.000051894	-0.000014032	0.000025947	-0.000007015	0.000103789	-0.000003508
21	0.000028779	-0.000007261	0.000014389	-0.00000363	0.000057559	-0.000001815
22	0.00001596	-0.000003757	0.00000798	-0.000001878	0.000031922	-0.000000939
23	0.000008851	-0.000001944	0.000004425	-0.000000971	0.000017704	-0.000000486
24	0.000004909	-0.000001006	0.000002454	-0.000000502	0.000009819	-0.000000251
25	0.000002722	-0.00000052	0.000001361	-0.000000259	0.000005446	-0.000000129
26	0.000001509	-0.000000269	0.000000754	-0.000000134	0.00000302	-0.000000066
27	0.000000837	-0.000000139	0.000000418	-0.000000069	0.000001675	-0.000000034
28	0.000000464	-0.000000071	0.000000231	-0.000000035	0.000000929	-0.000000017
29	0.000000257	-0.000000036	0.000000128	-0.000000018	0.000000515	-0.000000008
30	0.000000142	-0.000000018	0.000000071	-0.000000009	0.000000285	-0.000000004
31	0.000000078	-0.000000009	0.000000039	-0.000000004	0.000000158	-0.000000002
32	0.000000043	-0.000000004	0.000000021	-0.000000002	0.000000087	-0.000000001
33	0.000000023	-0.000000002	0.000000011	-0.000000001	0.000000048	0
34	0.000000012	-0.000000001	0.000000006	0	0.000000026	0
35	0.000000006	0	0.000000003	0	0.000000014	0
36	0.000000003	0	0.000000001	0	0.000000007	0
37	0.000000001	0	0	0	0.000000003	0
38	0	0	0	0	0.000000001	0
39	0	0	0	0	0	0

In Table 3 below, we choose

$$v_n^i = \begin{cases} 2ix_n, & x_n \in [-4,0] \\ \{-\frac{3x_n}{9i}\}, & x_n \in (0,10i], \end{cases}$$

to show that the convergence of the sequence is independent of the choice of $v_n^i \in T^i x_n$, if the common element is a strict fixed point for each T^i.

Table3. $v_n^i = 2ix_n, x_n \le 0,$ **otherwise** $\frac{-3x_n}{9i}$.

n	x_n	K_{n+1}	x_n	K_{n+1}
0	3	[-4, 10]	-3	[-4,10]
1	3	[-4, 1.635451761]	-3	[-1.553135775, 10]
2	1.635451761	[-4, 0.898951648]	-1.553135775	[-0.81095412, 10]
3	0.898951648	[-4, 0.49566239]	-0.81095412	[-0.425063676, 10]
4	0.49566239	[-4, 0.273744239]	-0.425063676	[-0.223287073, 10]
5	0.273744239	[-4, 0.151335613]	-0.223287073	[-0.117459451, 10]
6	0.151335613	[-4, 0.083720857]	-0.117459451	[-0.061850353, 10]
7	0.083720857	[-4, 0.046338328]	-0.061850353	[-0.03259221, 10]
8	0.046338328	[-4, 0.025657188]	-0.03259221	[-0.017184205, 10]
9	0.025657188	[-4, 0.014210344]	-0.017184205	[-0.009064388, 10]
10	0.014210344	[-4, 0.007872308]	-0.009064388	[-0.004783046, 10]
11	0.007872308	[-4, 0.004361977]	-0.004783046	[-0.002524647, 10]
12	0.004361977	[-4, 0.002417323]	-0.002524647	[-0.001332927, 10]
13	0.002417323	[-4, 0.001339817]	-0.001332927	[-0.000703891, 10]
14	0.001339817	[-4, 0.000742689]	-0.000703891	[-0.000371779, 10]
15	0.000742689	[-4, 0.00041173]	-0.000371779	[-0.000196397, 10]
16	0.00041173	[-4, 0.000228274]	-0.000196397	[-0.000103764, 10]
17	0.000228274	[-4, 0.000126571]	-0.000103764	[-0.000054829, 10]
18	0.000126571	[-4, 0.000070184]	-0.000054829	[-0.000028975, 10]
19	0.000070184	[-4, 0.00003892]	-0.000028975	[-0.000015313, 10]
20	0.00003892	[-4, 0.000021584]	-0.000015313	[-0.000008093, 10]
21	0.000021584	[-4, 0.00001197]	-0.000008093	[-0.000004277, 10]
22	0.00001197	[-4, 0.000006638]	-0.000004277	[-0.00000226, 10]
23	0.000006638	[-4, 0.000003681]	-0.00000226	[-0.000001194, 10]
24	0.000003681	[-4, 0.000002041]	-0.000001194	[-0.000000631, 10]
25	0.000002041	[-4, 0.000001132]	-0.000000631	[-0.000000333, 10]
26	0.000001132	[-4, 0.000000627]	-0.000000333	[-0.000000175, 10]
27	0.000000627	[-4, 0.000000347]	-0.000000175	[-0.000000092, 10]
28	0.000000347	[-4, 0.000000192]	-0.000000092	[-0.000000048, 10]
29	0.000000192	[-4, 0.000000106]	-0.000000048	[-0.000000025, 10]
30	0.000000106	[-4, 0.000000058]	-0.000000025	[-0.000000013, 10]
31	0.000000058	[-4, 0.000000032]	-0.000000013	[-0.000000006, 10]
32	0.000000032	[-4, 0.000000017]	-0.000000006	[-0.000000003, 10]
33	0.000000017	[-4, 0.000000009]	-0.000000003	[-0.000000001, 10]
34	0.000000009	[-4, 0.000000004]	-0.000000001	[0, 10]
35	0.000000004	[-4, 0.000000002]	0	

36	0.000000002	[-4, 0.000000001]
37	0.000000001	[-4, 0]
38	0	

Case ii.

Example 2. Assume the sequences and bifunctions to be as in Example 1 and define $T^i: \mathbb{R} \to P(\mathbb{R})$ by

$$T^i x = [x, x + i]$$

Clearly, $F_s(T^i) = \emptyset$ for each i but $\mathbb{F} = \bigcap_{i=1}^{N} F(T^i) \cap \ (\bigcap_{i=1}^{N} EP(f^i)) = \{0\} \neq \emptyset$. The numerical computation of the algorithm for the common element shows that the convergence of the sequence to a common element depends on the choice of $v_n^i \in T^i x_n$. Tables 4 and 5 are the results for $v_n^i \in P_{T^i} x_n$ and $v_n^i \notin P_{T^i} x_n$ respectively. We consider

(i) $v_n^i = x_n \in P_{T^i} x_n$

(ii) $v_n^i = x_n + i \notin P_{T^i} x_n$ for $x_n \in [0, \infty)$.

Table 4. $v_n^i = x_n \in P_{T^i} x_n$

n	x_n	K_{n+1}	x_n	K_{n+1}
0	3	[-∞,∞]	-3	[-∞, ∞]
1	3	[-∞,1.636363636]	-3	[-1.636363636, ∞]
2	1.636363636	[-∞, 0.899999999]	-1.636363636	[-0.899999999, ∞]
3	0.899999999	[-∞, 0.496551723]	-0.899999999	[-0.496551723, ∞]
4	0.496551723	[-∞, 0.274410162]	-0.496551723	[-0.274410162, ∞]
5	0.274410162	[-∞, 0.151801366]	-0.274410162	[-0.151801366, ∞]
6	0.151801366	-∞ [, 0.084032899]	-0.151801366	[-0.084032899, ∞]
7	0.084032899	[-∞, 0.046541297]	-0.084032899	[-0.046541297, ∞]
8	0.046541297	[-∞, 0.025786394]	-0.046541297	[-0.025786394, ∞]
9	0.025786394	[-∞, 0.014291254]	-0.025786394	[-0.014291254, ∞]
10	0.014291254	[-∞, 0.007922325]	-0.014291254	[-0.007922325, ∞]
11	0.007922325	[-∞, 0.004392576]	-0.007922325	[-0.004392576, ∞]
12	0.004392576	[-∞, 0.002435883]	-0.004392576	[-0.002435883, ∞]
13	0.002435883	[-∞, 0.001350993]	-0.002435883	[-0.001350993, ∞]
14	0.001350993	[-∞, 0.000749378]	-0.001350993	[-0.000749378, ∞]
15	0.000749378	[-∞, 0.000415713]	-0.000749378	[-0.000415713, ∞]
16	0.000415713	[-∞, 0.000230635]	-0.000415713	[-0.000230635, ∞]
17	0.000230635	[-∞, 0.000127965]	-0.000230635	[-0.000127965, ∞]
18	0.000127965	[-∞, 0.000071004]	-0.000127965	[-0.000071004, ∞]
19	0.000071004	[-∞, 0.000039401]	-0.000071004	[-0.000039401, ∞]
20	0.000039401	[-∞, 0.000021865]	-0.000039401	[-0.000021865, ∞]
21	0.000021865	[-∞, 0.000012134]	-0.000021865	[-0.000012134, ∞]
22	0.000012134	[-∞, 0.000006734]	-0.000012134	[-0.000006734, ∞]
23	0.000006734	[-∞, 0.000003737]	-0.000006734	[-0.000003737, ∞]
24	0.000003737	[-∞, 0.000002074]	-0.000003737	[-0.000002074, ∞]

25	0.000002074	[-∞, 0.000001151]	-0.000002074	[-0.000001151, ∞]
26	0.000001151	[-∞, 0.000000638]	-0.000001151	[-0.000000638, ∞]
27	0.000000638	[-∞, 0.000000354]	-0.000000638	[-0.000000354, ∞]
28	0.000000354	[-∞, 0.000000196]	-0.000000354	[-0.000000196, ∞]
29	0.000000196	[-∞, 0.000000108]	-0.000000196	[-0.000000108, ∞]
30	0.000000108	[-∞, 0.000000059]	-0.000000108	[-0.000000059, ∞]
31	0.000000059	[-∞, 0.000000032]	-0.000000059	[-0.000000032, ∞]
32	0.000000032	[-∞, 0.000000017]	-0.000000032	[-0.000000017, ∞]
33	0.000000017	[-∞, 0.000000009]	-0.000000017	[-0.000000009, ∞]
34	0.000000009	[-∞, 0.000000004]	-0.000000009	[-0.000000004, ∞]
35	0.000000004	[-∞, 0.000000002]	-0.000000004	[-0.000000002, ∞]
36	0.000000002	[-∞, 0.000000001]	-0.000000002	[-0.000000001, ∞]
37	0.000000001	[-∞, 0]	-0.000000001	[0, ∞]
38	0	[-∞, 0]	0	

Table 5. $\bigcap_{i=1}^{4} K_8^i = \emptyset$

i	$K_8^i,\ x_0 = 3$	$K_8^i,\ x_0 = -3$
	$[-4, -0.049609123]$	$[0.049609123, 10]$
	$[-4, -0.028650291]$	$[0.028650291, 10]$
	$[-4, -0.015816393]$	$[0.015816393, 10]$
	$[-0.007613719, 10]$	$[-4, 0.007613719]$

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